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A Generalized Framework for Linearly-Constrained Singularity-Free Control Moment Gyro Steering Laws

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Constrained steering laws have no inherent torque error and do not require pre-computed CMG paths while maintaining singularity-free motion, three advantageous characteristics that make this type of steering law worthy of further study. However, in order for constraint-based steering laws to ideally solve the SGCMG steering problem, it must be generalized to be applicable to non-specific geometries. This paper proposes a generalized framework for steering laws that are explicitly linear with respect to the gimbal rates. This formulation is followed by a discussion of characteristics of constraints used in this law, and general principles for designing a singularity-free constraint. A simple example using scissored-pair SGCMGs is used to demonstrate the principles behind this generalized framework. Finally, the theoretical basis for a new set of steering laws is proposed, with emphasis on the analytical basis for such a law using the insight gained from the generalized formulation.

Nomenclature

A	=	system matrix augmented with constraint law
α	=	a general scalar constant
β_1	=	a scaling value for the first row of the Jacobian in the constraint equation
β_2	=	a scaling value for the second row of the Jacobian in the constraint equation
β_c	=	a non-zero scaling value for the cross-product component of the constraint equation
C	=	mxn matrix of general constraint equations
D	=	mx1 matrix of the solution to the constraint equations
d	=	a scalar solution to a constraint equation
Δ	=	the determinant of the A system matrix
h	=	the magnitude of the angular momentum of an individual CMG
J	=	3xn system Jacobian matrix
n	=	number of SGCMGs in an array
m	=	number of constraints in a constraint-based steering law
Φ	=	nx1 matrix of gimbal angles (made up of ϕ_i)
$\dot{\Phi}$	=	nx1 matrix of gimbal rate commands (made up of $\dot{\phi}_i$)
ϕ	=	individual CMG gimbal angles
τ_C	=	matrix of torques values commanded by spacecraft
τ_j	=	scalar of individual torque components (in the body frame)

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I. Introduction

CONTROL moment gyroscopes (CMGs) are momentum exchange actuators used to control the attitude of a spacecraft bus that are particularly useful in applications requiring high slewing rates or large torques. Although several types of CMGs exist, including variable speed CMGs[1], and double-gimbal CMGs[2], the single-gimbal CMG (SGCMG) has a combination of cost-effectiveness and mechanical simplicity that makes it an attractive choice for implementation on space systems[3]. Since SGCMGs only gimbal about one axis, a minimum of three are necessary to achieve full attitude control, although it is more common to use an array of four or more. The CMG array is controlled by a steering law, which dictates how the CMGs move within the array (usually via gimbal rates) to provide the required torque.

The greatest drawback to this attitude control system is the presence of kinematic singularities at certain gimbal configurations. These singularities are points at which the CMG array is incapable of producing torque in a particular direction, which often results in an error in the spacecraft attitude. A major research focus has been designing steering laws for SGCMG arrays such that the system is capable of handling these singularities while maintaining a level of practicality and applicability to actual space systems.

One class of CMG steering laws uses linear constraints (whether in hardware or software) to avoid singularities while finding a solution in-situ without induced errors. Despite the fact that several variations of this steering law exist in the literature[4,5], a generalized form of this particular CMG steering law has yet to be presented. Once in a generalized form, the problem is freed from specific array geometries or constraint laws, which frees the design space to allow for optimized configurations. This paper presents a generalized mathematical description of steering laws with linear constraints and uses an example of a scissored-pair CMGs to demonstrate the validity of the formulation. The paper then discusses a unique constraint law designed by the authors based on the principles suggested by the generalized formulation.

II. Background and Context for CMG Steering Laws

CMGs have been researched as spacecraft actuators for nearly four decades[6] and an extensive portion of this research has been dedicated to finding an optimal way in which to steer SGCMG arrays. Standard approaches to this problem can be sorted into roughly five categories (according to the categories in Kurokawa's steering law survey)[7]: the Moore-Penrose pseudo-inverse steering law, inexact singularity robust steering laws, offline-planning solutions, gradient or null motion methods, and restricted workspace/constrained steering laws. Each method has a different set of advantageous characteristics (shown in Table 1), but as Kurokawa concludes[7], a perfect solution has yet to be found.

Although it is possible that a perfect SGCMG steering law does not exist, establishing the characteristics of such an ideal illuminates the deficiencies of existing laws, and may help direct further research efforts to finding an optimal result. Thus, by cross-referencing the desired capabilities of a steering law from multiple references, six distinct characteristics of an ideal SGCMG steering law become apparent:

1) Capable of handling singularities
2) Exact (error-free)
3) Instantaneous
4) Technologically feasible
5) Cost-effective (high performance to cost ratio)
6) General with regard to maneuver and CMG configuration

The most obvious requirement for a SGCMG steering law is the ability to cope with kinematic singularities. Steering logics that prevent the CMGs from encountering singularities at all are broadly referred to as "singularity-avoidance" laws, whereas those that are designed to enable the array to pass through singularities are called "singularity-robust" laws. Steering laws that do not address how to handle singularities, such as the Moore-Penrose pseudo-inverse[8], are generally not considered for actual implementation on spacecraft; instead, they may provide a framework for developing steering laws that can then be modified to handle singularities.

Since CMGs are particularly well-suited to highly agile spacecraft, the applications for which most CMG arrays are being considered have high precision requirements for attitude control. Thus, while a large category of CMG steering logics intentionally add errors to the solution as a relatively quick way to sidestep singularities[9], it is preferable for the steering law to produce exact solutions.

An optimal steering law must also be capable of interpreting a torque command and implement it in-situ on the spacecraft. The off-line approaches to developing singularity-free paths for the CMG array are generally lengthy computationally-intensive modeling[11] and cannot account for all of the errors that are inevitably present in any

Table 1. Summary of General Steering Law Characteristics. *This table matches the basic steering law categories[7] with their associated "ideal" characteristics. Y implies that most specific laws in that category are described by the characteristic corresponding to that column. N implies that the associated steering law group is not typified by the characteristic in that column. S implies that the characteristic describes some specific laws in that category, but it is not a general trend for that category. Note that these are generalizations and that exceptions may exist. Also, this chart only considers some of the most popular SGCMG steering law categories- it is not exhaustive.*

Steering Law	Examples	Singularity Free (1)	Exact (2)	Instantaneous (3)	Technologically Feasible (4)	Cost-Effective (# CMGs / Maximal Use of Workspace) (5)	Generalized (6)
Moore-Penrose Pseudo-Inverse	Pseudo-Inverse[3]	N	Y	Y	N	3 / Y	Y
Allowing Torque Error	SR Inverse[8]; Singular Direction Avoidance[10], Off-Diagonal SR Inverse[12]	Y	N	Y	S	3 / Y	Y
Offline Planning	Optimal State for SR Inverse[11], Plan entire satellite motion	Y	Y	N	S	3 / Y	Y
Gradient / Null-Motion	Various[9,12-14], Honeywell Patent[16]	Y	Y	Y	S	4 -6 / N	N
Restricted Workspace / Constraint-Based	Scissored-pairs[4], Kurokawa's[15]	Y	Y	Y	S	4-6 / N	N

spacecraft maneuver. Instantaneous steering laws are ideal because they are less reliant on a priori knowledge, consequently leading to a more responsive spacecraft.

Ideally, a steering law does not make infeasible demands on the hardware of the system. This characteristic is usually violated by the exceptionally high gimbal rates that are required in certain types of null-motion and singularity-avoidance steering laws. Proving that a steering law is technologically feasible can be done analytically or in simulation, but is ultimately necessary for the perfect solution.

Any steering law that classifies as ideal must be shown to provide a reasonable amount of performance relative to its size, weight, and other cost factors. Methods that severely constrain the momentum envelope of the CMG array or require large numbers of CMGs to be effective are generally not ideal for this reason. The chart in Table 1 shows the minimum number of CMGs needed to operate the steering law (smaller is better for cost effectiveness), and indicates whether the workspace volume is lost due to the constraints.

Finally, an ideal steering law is applicable to a CMG array regardless of the particular maneuvers commanded by the spacecraft or the physical configuration of the array. With a general steering law, the spacecraft motion and CMG array design are a design parameter selected based on the mission requirements rather than a constraint on the spacecraft's capabilities. A general steering law would also be robust with regards to CMG failures that change the geometry of the available array of actuators. In practice, however, it is more straightforward to design CMG steering laws for specific geometries such as the roof-top array[14], the pyramid array[7], and multiparallel systems[15]. Many gradient methods and constraint-based steering laws suffer from a lack of generality, preventing their wider application to the CMG field.

Of the six ideal characteristics, the first three are intrinsic to the steering law's method and can therefore not be improved if the method does not have that characteristic. For example, steering laws that allow for torque error are by definition not exact. While researchers continue to refine these methods to reduce the effect of the induced errors on the output of the array, this method will never be exact. Thus, since null motion and constraint-based methods are the only categories that definitely have the first three ideal characteristics, the "ideal" steering law is most likely going to be found within these groups.

The remainder of this paper examines the restricted workspace/ constraint-based steering laws for linear constraints. It should be noted that constraints do not always have to be linear with respect to the gimbal rates, as studied by Kurokawa's research in references 5 and 16. Section III describes a general form for these steering laws and establishes special cases of this form. Section IV contains simulation results for a unique linearly-constrained steering law designed for Cornell's nanosatellite CMG testbed.

III. Generalized Form for Linearly-Constrained Steering Laws

A. Description

Standard CMG steering laws are designed to determine the gimbal rates necessary for the CMG array to provide the commanded amount of torque in order to maneuver the spacecraft in the desired manner. A typical description of these commanded torques in terms of the gimbal rates is shown in Equation 1, which nearly all common CMG steering laws solve using a version of the Moore-Penrose pseudo-inverse solution in Equation 2:

$$\tau_C = J(\Phi)\dot{\Phi} \tag{1}$$

$$\dot{\Phi}(\tau) = \left(J(\Phi)^T J(\Phi)\right)^{-1} J(\Phi)^T \tau_C \tag{2}$$

where J is the Jacobian of the CMG array's total momentum as a function of the gimbal angles, τ_c is the vector of torques commanded by the spacecraft and $\dot{\Phi}$ is the nx1 array of gimbal rates required to achieve that torque. Linearly constrained steering laws are no exception; however, in addition to describing the dynamics of the problem via the Jacobian, these steering laws also include external constraints on the problem. Equation 3 is a general description of a set of linear constraint equations:

$$D = C \cdot \dot{\Phi} \tag{3}$$

Thus, the Equation 1 can be augmented to include the constraints as shown in Equation 4:

$$\begin{bmatrix} \tau_C \\ D \end{bmatrix} = \begin{bmatrix} J(\Phi) \\ C \end{bmatrix} \dot{\Phi} = A\dot{\Phi} \tag{4}$$

By substituting the matrix A into the space once occupied by the Jacobian in the pseudo-inverse equation, the linearly constrained steering laws can be generally described as shown in Equation 5:

$$\dot{\Phi}(\tau) = \left(A(\Phi)^T A(\Phi)\right)^{-1} A(\Phi)^T \tau_C \tag{5}$$

This mathematical description of linearly constrained steering laws provides a method of generalizing all variations of this class of steering laws. By providing a general description of this category of steering laws, the array geometry can be selected based on mission requirements, not steering law constraints, making linearly constrained steering laws one step closer to an ideal steering logic.

B. Observations on Constraint Design

The nature of the constraints described above merits further scrutiny, since these additional equations provide a basis for the singularity avoidance properties of the steering laws involved. Firstly, holomonic constraints (which dictate the gimbal angles directly) can be incorporated into this formulation by simply taking the derivative of the constraint involved. The scissored-pair constraint is one such example that is discussed in further detail later in the paper. However, this method does restrict the initial conditions such that the array does not start out singular. Also, although the possibility is not explored in this paper, this formulation may lend itself to non-holonomic constraints due to the fact that the constraints only need to be a function of the gimbal rates (independent of specific sets of gimbal angles).

Secondly, the number of constraints is not fixed. It is conceivable that an array of SGCMGs is capable of producing singularity-free motion even with an unconstrained degree of freedom. However, for simplicity, the examples in the remainder of this paper only consider fully constrained systems where the number of constraints m must be equivalent to the number of excess SGCMGs in operation (n – dimension of system). (Note that constraint-based steering laws assume that $n >$ dimension of the system in order to exploit the null space of the system). For example, controlling a 2-dimensional momentum space with three SGCMGs will require one constraint. A fully constrained system produces a square A matrix, which in turn means that the pseudo-inverse in Equation 5 can be replaced with a direct inverse.

It is also worth noting that in order for a constraint-based steering law to operate properly, the constraint must be properly enforced if implemented at a software level. Thus, some minor amount of low-bandwidth feedback may be required to avoid an accumulation of numerical error that would then violate the constraint.

Finally, the generalized form of the augmented matrix shown in Equation 4 suggests that in order for the A matrix to be non-singular, it is essential for the constraint matrix C to contain rows that are linearly independent of the original Jacobian J of the system. When a fully-constrained system has constraints that are guaranteed to maintain linearly independence for the range of possible motions, the array becomes singularity-free. Thus, it is possible to write constraint equations with some component in a direction perpendicular to the directions described by the rows of the Jacobian, and thereby guarantee the system does not encounter an internal singularity.

C. Example: Scissored-Pair CMGs

To demonstrate how this general formulation applies to specific examples, it is illustrative to examine the special case of a simple, well-established, linearly constrained steering law known as a "scissored pair" arrangement.[4] In a scissored pair configuration, two SGCMGs are constrained such that their gimbal angles are equal and opposite ($\phi_1 = -\phi_2$). Alternatively, the angles can be constrained to be the same ($\phi_1 = \phi_2$), but the gimbal axes are exactly opposite. The "scissoring" motion caused by this constraint produces an output torque in a direction fixed along the reflection line between the two CMGs, as shown in Figure 1. Thus, complete six degree of freedom control of a spacecraft requires three scissored-pairs of SGCMGs. One potential arrangement is shown in Figure 2.

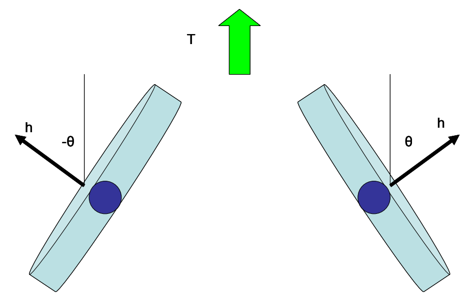

Figure 1. Scissored Pair Constraint. *The scissored pair CMGs are constrained such that the two CMGs have equal and opposite gimbal angles, which results in a total torque output along a single axis.*

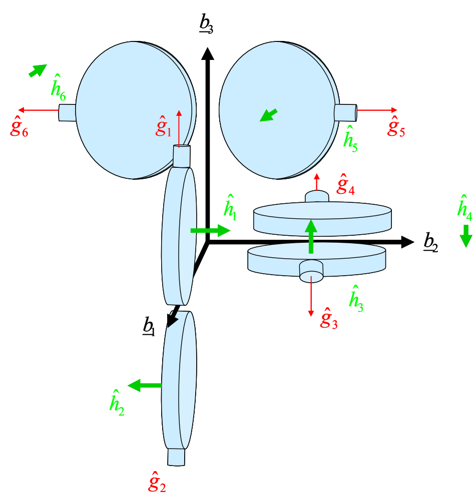

Figure 2. Arrangement of 3 Scissored-Pairs. *This figure demonstrates one possible arrangement of the three scissored pairs of CMGs that are required for 6-DOF control of a spacecraft.*

Scissored-pair arrangements of SGCMGs are singularity-free except at the saturation singularities where the pair is producing the maximum torque it is capable of producing in one direction. However, in order for a scissored pair arrangement to be implemented as the primary method of control on a spacecraft, a minimum of six CMGs is required, with each CMG operating within a reduced envelope of its effective momentum workspace. Thus, this configuration of CMGs is generally not considered cost-effective despite its singularity avoidance properties and its relative simplicity.

The Jacobian of the six CMGs arranged as in Figure 2 (six CMGs aligned along the three Cartesian axes in pairs, with opposite gimbal angles and without assuming a scissored-pair constraint) can be computed to be:

$$J_{ScissoredPair} = h\begin{bmatrix} -\cos\phi_1 & -\cos\phi_2 & 0 & 0 & -\sin\phi_5 & \sin\phi_6 \\ -\sin\phi_1 & \sin\phi_2 & -\cos\phi_3 & -\cos\phi_4 & 0 & 0 \\ 0 & 0 & -\sin\phi_3 & \sin\phi_4 & -\cos\phi_5 & -\cos\phi_6 \end{bmatrix} \quad (6)$$

In order to make the example arrangement into a scissored-pair set up, the pairs on each axis must be constrained to operate such that each pair always has the same gimbal angles. In the example given in Figure 2 (where the gimbal axes are opposite for each pair), this set of constraints can be expressed as: $\phi_1 - \phi_2 = 0;\ \phi_3 - \phi_4 = 0;\ \phi_5 - \phi_6 = 0$. Taking the derivative of these holonomic constraints produces a similar constraint on the gimbal rates. When placed in a matrix form, it produces Equation 7.

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}\dot{\Phi} \quad (7)$$

This constraint equation now fits the form of the *C* and *D* matrices described in the general formulation for linearly constrained steering laws, with the values of each defined in Equation 8.

$$D = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}; C = \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \quad (8)$$

Augmenting the Jacobian and constraint matrices together, the final formulation of a scissored pair arrangement of CMGs is shown in Equation 9.

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ \tau_3 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -h\cos\phi_1 & -h\cos\phi_2 & 0 & 0 & -h\sin\phi_5 & h\sin\phi_6 \\ -h\sin\phi_1 & h\sin\phi_2 & -h\cos\phi_3 & -h\cos\phi_4 & 0 & 0 \\ 0 & 0 & -h\sin\phi_3 & h\sin\phi_4 & -h\cos\phi_5 & -h\cos\phi_6 \\ 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix}\dot{\Phi} \quad (9)$$

The singularity avoidance properties of this constraint-based steering law can be seen by the fact that the rows of the constraint matrix in Equation 8 are always linearly independent of the original Jacobian in Equation 6. In this case, the linear independence of these rows can be demonstrated by the dot product, as shown in Equations 10a – 10c. By plugging in the fact that the paired gimbal angles must be equivalent to one another (Equation 10a and 10b), we can easily see that the constraint is perpendicular to the vectors in the Jacobian, making it linearly independent. This process can be repeated for the remainder of the constraints to show that the scissored-pair arrangement has no internal singularities.

$$\begin{aligned} C(1,:)\bullet J(1,:) &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}\bullet\begin{bmatrix} -h\cos\phi_1 & -h\cos\phi_2 & 0 & 0 & -h\sin\phi_5 & h\sin\phi_6 \end{bmatrix} \\ &= -h\cos\phi_1 + h\cos\phi_2 = 0 \end{aligned} \quad (10a)$$

$$\begin{aligned} C(1,:)\bullet J(2,:) &= \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}\bullet\begin{bmatrix} -h\sin\phi_1 & h\sin\phi_2 & -h\cos\phi_3 & -h\cos\phi_4 & 0 & 0 \end{bmatrix} \\ &= -h\sin\phi_1 - h\sin\phi_2 = 0 \end{aligned} \quad (10b)$$

$$C(1,:)\bullet J(3,:)=\begin{bmatrix}1 & -1 & 0 & 0 & 0 & 0\end{bmatrix}\bullet\begin{bmatrix}0 & 0 & -h\sin\phi_3 & h\sin\phi_4 & -h\cos\phi_5 & -h\cos\phi_6\end{bmatrix} \\ =0 \tag{10c}$$

IV. Triplet Steering Law

A. Theoretical Discussion

This general formulation of linearly constrained steering laws provides a framework for understanding the principles behind this type of CMG steering. For simplicity, first consider the case of a planar CMG momentum envelope (where the gimbal axes of the CMGs are all aligned with one another). A minimum of two CMGs are required in order to produce any torque within saturation limits in a two-dimensional plane, as shown in Figure 3. (One CMG only allows for a one-dimensional momentum manifold, since the magnitude remains constant for all possible angles of the momentum vector relative to the body axes.) A set of two CMGs has a deterministic solution to each torque command (i.e., only one pair of gimbal angles corresponds to the appropriate torque, regardless of which CMG has which specific gimbal angle).

However, if three CMGs are placed in a plane (a "triplet" set), the additional CMG adds a degree of freedom, making the torque commands non-deterministic (for example, an infinite number of gimbal angles relative to the body axes can correspond to the zero momentum state, as shown in Figure 3). This additional degree of freedom introduces an internal singularity at a ring 1-*h* in radius around the center of the axes, corresponding to a situation where two momentum vectors exactly cancel out and all three vectors are collinear. By having three CMGs, as discussed above, one constraint equation is required to fully constrain the system in a two-dimensional workspace. This constraint can be designed to successfully avoid the internal singularity while also deterministically providing the required torque by exploiting the null space of the Jacobian. The constraint equation that fills this need must accomplish three objectives: 1) avoid the singularity state, 2) capture the maximum possible range of momentum values and 3) include the zero-momentum case.

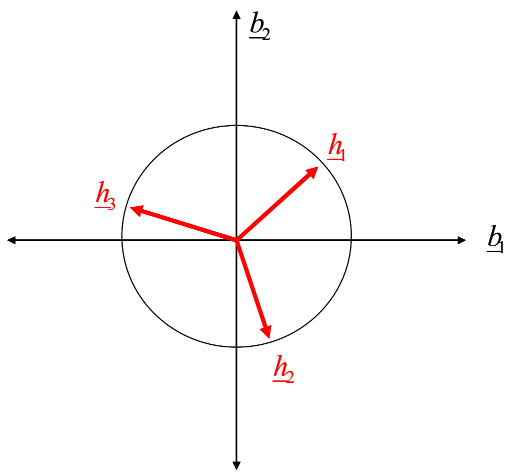

Figure 3. Planar Representation of CMG Momentum. *The angular momentum vectors of three planar CMGs span the two-dimensional momentum manifold in a non-deterministic way. This configuration shows one possible zero-momentum state for the array.*

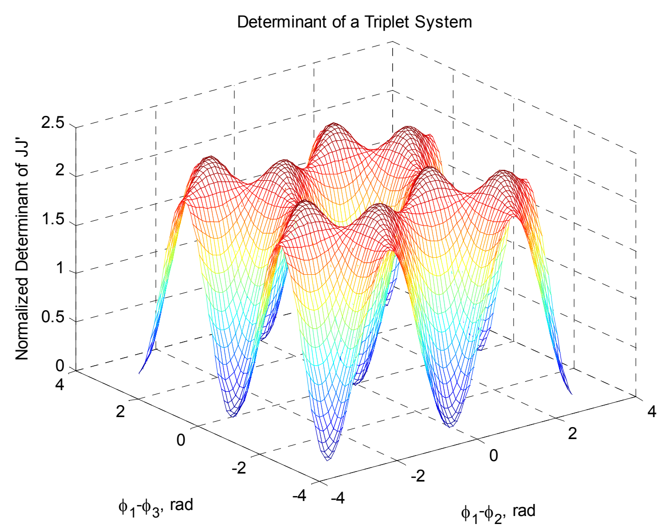

Figure 4. Triplet Singularity Plot. *This plot shows the determinant of the JJ' product as a function of the relative gimbal angles in the triplet. A zero determinant represents a singularity. The extra degree of freedom in the problem – the third CMG – allows a constraint equation to be written such that the steering law avoids these wells while still producing the commanded torque.*

The triplet array also has a unique representation: because the internal singularities occur at specific relative gimbal angles, the array can be represented by two coordinates instead of three. Although this does not change the dynamics of the system, it simplifies the representation and may result in faster numerical computations in implementation. It also provides greater insight into the geometry of the problem, because such a system can be rendered in three-dimensional space, as shown in Figure 4, which plots the determinant of triplet system as a function of relative gimbal angles.

The mathematics of this problem (assuming that all of the gimbal axes are in the positive $\underline{\hat{b}}_3$ direction and the angular momentum points in the positive $\underline{\hat{b}}_1$ direction at $\phi = 0$) in terms of the general formulation given in section III involve a 2x 3 Jacobian as shown in Equation 11.

$$J_{Planar} = \begin{bmatrix} -\sin\phi_1 & -\sin\phi_2 & -\sin\phi_3 \\ \cos\phi_1 & \cos\phi_2 & \cos\phi_3 \end{bmatrix} = \begin{bmatrix} J_1 \\ J_2 \end{bmatrix} \tag{11}$$

where J_1 and J_2 are simply the top and bottom rows of the planar Jacobian, respectively.

Singularities occur when the matrix being inverted in the Moore-Penrose Pseudo-Inverse (Equation 2), in this case, the augmented *A* matrix as described in the general formulation for this problem, is no longer full rank. In order to ensure singularity avoidance, it is sufficient to write a constraint equation that is guaranteed to be linearly independent of the two existing rows of the Jacobian. If the cross-product of two vectors is non-zero, the two vectors are linearly independent. Thus, a singularity-free constraint equation must include a scaled component in the direction of the cross-product of the two rows of the Jacobian. Equation 12 shows the constraints written in the form of the general linearly constrained steering law from section III.

$$D = d = \left[\beta_1 J_1 + \beta_2 J_2 + \beta_c \left(J_1 \times J_2\right)\right]\dot{\Phi} = C\dot{\Phi} \tag{12}$$

where *d* is a scalar solution to the constraint equation, β_1 and β_2 are scaling values for the component of the vector in the direction of the Jacobian rows, and β_c is a non-zero scaling value for the constraint equation. Also, note that since the cross-product term is necessarily non-zero, if β_1 and β_2 are 0, *d* cannot also be zero. Equation 13 shows the general form of this expression in the structure developed in section III.

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ d \end{bmatrix} = \begin{bmatrix} -\mathrm{s}\phi_1 & -\mathrm{s}\phi_2 & -\mathrm{s}\phi_3 \\ \mathrm{c}\phi_1 & \mathrm{c}\phi_2 & \mathrm{c}\phi_3 \\ -\beta_1\,\mathrm{s}\phi_1 + \beta_2\,\mathrm{c}\phi_1 \ldots & -\beta_1\,\mathrm{s}\phi_2 + \beta_2\,\mathrm{c}\phi_2 \ldots & -\beta_1\,\mathrm{s}\phi_3 + \beta_2\,\mathrm{c}\phi_3 \ldots \\ \ldots + \beta_c\left(-\mathrm{s}\phi_2\mathrm{c}\phi_3 + \mathrm{s}\phi_3\mathrm{c}\phi_2\right) & \ldots + \beta_c\left(-\mathrm{s}\phi_3\mathrm{c}\phi_1 + \mathrm{s}\phi_1\mathrm{c}\phi_3\right) & \ldots + \beta_c\left(-\mathrm{s}\phi_1\mathrm{c}\phi_2 + \mathrm{s}\phi_2\mathrm{c}\phi_1\right) \end{bmatrix}\dot{\Phi} \tag{13}$$

B. Expansion to 6-DOF

This triplet analysis can be expanded to six degrees of freedom by recognizing that two sets of triplets (six SGCMGs total) positioned such that the momentum envelopes are at a perpendicular offset relative to one another can be used together for spacecraft attitude control. This type of configuration would result in complete singularity-free control over two planes at an angle to one another, and thus would span the three-dimensional range of momentum values. One such configuration is shown in Figure 6. The 6-DOF triplet steering law is by no means an ideal steering law as described previously, largely because it is only applicable to a particular type of array geometry, and requires a relatively large number of CMGs to operate. However, it is instructive to study a particular sub-

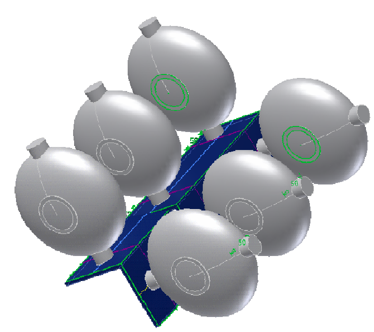

Figure 6. Two Triplets of CMGs at Right Angles. *In order to expand a triplet analysis to six degrees of freedom, two sets of triplets at some angle relative to one another would be required..*

case of a constrained steering law in terms of the proposed general formulation, and use the insight of such a generalized formula to design steering laws from first principles, which may lay the foundation for future, more ideal steering laws.

A 6-DOF triplet steering law can be augmented as two triplet sets, as described in Equation 14:

$$\begin{bmatrix} \tau_1 \\ \tau_2 \\ d_1 \\ \tau_3 \\ \tau_4 \\ d_2 \end{bmatrix} = \begin{bmatrix} & & & 0 & 0 & 0 \\ & A_1 & & 0 & 0 & 0 \\ & & & 0 & 0 & 0 \\ 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & A_2 & \\ 0 & 0 & 0 & & & \end{bmatrix} \dot{\Phi} \tag{14}$$

where A_1 represents the augmented matrix from the first triplet set (of the form shown in Equation 13), and A_2 represents the augmented matrix from the second triplet set. The three-dimensional torque vector will be a combination of the τ_{1-4}, depending on the orientation of the triplet sets relative to the body frame of the spacecraft. By controlling singularity-free subsets of the overall CMG array, the steering law only needs to determine how to distribute the torque command between the two subsets, and the resulting motion should be guaranteed singularity-free.

V. Conclusions

Reduced workspace/constrained steering laws provide an error-free, instantaneous method for directing CMG arrays while avoiding singularities but suffer from reduced cost-effectiveness and a lack of generality with regards to particular CMG array configurations. A general mathematical framework for describing the steering laws with constraint equations that are linear with respect to the gimbal rates is described. This formulation suggests that by choosing linear constraints that lie in a space perpendicular to the Jacobian, the augmented system matrix A can be made singularity free. This fact is demonstrated with a simple scissored-pair array example. Taking this idea one step further, it is possible to define two-dimensional singularity-free momentum envelopes with a triplet set of SGCMGs. These triplets have unique properties that make defining the constraint a function of the rows of the system Jacobian and scaling values. When applied to a three-dimensional system, the triplets can also be paired to span all six degrees of freedom, with the steering law being responsible for distributing the torque between the two singularity-free triplet arrays.

The triplet analysis bears further study, particularly with regard to numerical simulations of the effect of various scaling parameters and constraint solutions. Further work should investigate the possibility of using non-holonomic constraints in the context of the generalized formulation, and extensions of this work should attempt to describe a generalized formulation for all constraint-based methods, even those not linear with respect to the gimbal rates. Once this is developed, a study of various CMG configurations can be performed to determine which geometries provide the best balance of cost-effectiveness and technological feasibility.

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References

[1] Schaub, H., and Junkins, J. "Singularity Avoidances Using Null Motion and Variable Speed Control Moment Gyros" *Journal of Guidance Navigation and Control*, Vol. 23, No. 1, Jan.-Feb. 2000, pp. 11-16.

[2] Ahmed, J. and Bernstein, D. "Adaptive Control of Double-gimbal Control-Moment Gyro with Unbalanced Rotor" *Journal of Guidance, Control, and Dynamics*, Vol. 25, No. 1, Jan.-Feb. 2002, pp. 105-115.

[3] Oh, H., and Vadali, S. "Feedback Control and Steering Laws for Spacecraft Using Single Gimbal Control Moment Gyros" *Journal of the Astronautical Sciences*, Vol 39, No. 2, April-June 1991, pp. 183-203.

[4] Cunningham D. and Driskill, G. "A Torque Balance Control Moment Gyroscope Assembly for Astronaut Maneuvering" NASA 19720018745, 1972.
[5] Kurokawa, H. "Constrained Steering Law of Pyramid-Type Control Moment Gyros and Ground Tests" *Journal of Guidance, Control, and Dynamics*, Vol 20, No. 3, May-June 1997, pp. 445-449.
[6] Havill, J. and Ratfill, J. "A Twin-Gyro Attitude Control System for Space Vehicles" *National Aeronautics and Space Administration*, 1964.
[7] Kurokawa, H. "Survey of Theory and Steering Laws of Single-Gimbal Control Moment Gyros" *Journal of Guidance, Control, and Dyamics*, Vol. 30, No. 5, Sept.-Oct. 2007, pp. 1331-1340.
[8] Farmer, J."A Reactive Torque Control Law for Gyroscopically Controlled Space Vehicles" NASA TM X- 64790, 1973.
[9] Cornick, D."Singularity Avoidance Control Laws for Single Gimbal Control Moment Gyros" *Guidance and Control Conference*, Boulder, NASA TM X- 64790, 1979.
[10] Ford, K. and Hall, C., "Singular Direction Avoidance Steering for Control-Moment Gyros" *Journal of Guidance, Control and Dynamics*, Vol. 23, No. 4, July – Aug. 2000, pp 648 – 656.
[11] Paradiso, J."Global Steering of Single Gimballed Control Moment Gyroscopes Using a Directed Search" *Journal of Guidance, Control, and Dynamics*, Vol. 15, No. 5, 1992, pp. 1236-1244.
[12] Hefner, R., and McKenzie, C., "A Technique for Maximizing the Torque Capability of Control Moment Gyro Systems," *Astrodynamics*, Vol. 54, No. AAS 83-387, 1983, pp. 905–920.
[13] Wie, B., "New Singularity Escape/Avoidance Steering Logic for Control Moment Gyro Systems," *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Austin, TX, 2003.
[14] Yoshikawa, T."Steering Law for Roof Type Configuration Control Moment Gyro System" *Automatica,* Vol. 13, 1979, pp. 359 -368.
[15] M. R. Elgersma, D. P. Johnson, M. A. Peck, B. K. Underhill, G. Stein, B. G. Morton, and B. J. Hamilton, "Method and system for controlling sets of collinear control moment gyroscopes," U.S. Patent 20070124032, Nov. 30, 2005.
[16] Kurokawa, H. "Exact Singularity Avoidance Control of the Pyramid Type CMG System" *Proceedings of the AIAA Guidance, Navigation, and Control Conference*, Washington D.C., 1994, pp. 170-180.